Searching for Universal Truths

Abstract Measure Theory

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Navigating Mathematical and Statistical Territories

Notations

- sets of numbers
  - **N** - set of natural numbers
  - **Z** - set of integers
  - $\mathbf{Z}_+$ - set of nonnegative integers
  - **Q** - set of rational numbers
  - **R** - set of real numbers
  - $\mathbf{R}_+$ - set of nonnegative real numbers
  - $\mathbf{R}_{++}$ - set of positive real numbers
  - **C** - set of complex numbers
- sequences $\langle x_i \rangle$ and the like
  - finite $\langle x_i \rangle_{i=1}^n$, infinite $\langle x_i \rangle_{i=1}^\infty$ - use $\langle x_i \rangle$ whenever unambiguously understood
  - similarly for other operations, *e.g.*, $\sum x_i$, $\prod x_i$, $\cup A_i$, $\cap A_i$, $\bigtimes A_i$
  - similarly for integrals, *e.g.*, $\int f$ for $\int_{-\infty}^{\infty} f$
- sets
  - $\tilde{A}$ - complement of A

  - $A \sim B$ - $A \cap \tilde{B}$
  - $A \Delta B$ - $(A \cap \tilde{B}) \cup (\tilde{A} \cap B)$
  - $\mathcal{P}(A)$ - set of all subsets of A
- sets in metric vector spaces
  - $\overline{A}$ - closure of set A
  - A° - interior of set A
  - $\mathbf{relint}\, A$ - relative interior of set A
  - $\mathbf{bd}\, A$ - boundary of set A
- set algebra
  - $\sigma(\mathcal{A})$ - σ-algebra generated by $\mathcal{A}$, *i.e.*, smallest σ-algebra containing $\mathcal{A}$
- norms in $\mathbf{R}^n$
  - $\|x\|_p$ $(p \geq 1)$ - p-norm of $x \in \mathbf{R}^n$, *i.e.*, $(|x_1|^p + \cdots + |x_n|^p)^{1/p}$
  - *e.g.*, $\|x\|_2$ - Euclidean norm
- matrices and vectors
  - a_i - i-th entry of vector a
  - A_{ij} - entry of matrix A at position (i, j), *i.e.*, entry in i-th row and j-th column
  - $\mathbf{Tr}(A)$ - trace of $A \in \mathbf{R}^{n \times n}$, *i.e.*, $A_{1,1} + \cdots + A_{n,n}$

- symmetric, positive definite, and positive semi-definite matrices
  - $\mathbf{S}^n \subset \mathbf{R}^{n\times n}$ - set of symmetric matrices
  - $\mathbf{S}^n_+ \subset \mathbf{S}^n$ - set of positive semi-definite matrices; $A \succeq 0 \Leftrightarrow A \in \mathbf{S}^n_+$
  - $\mathbf{S}^n_{++} \subset \mathbf{S}^n$ - set of positive definite matrices; $A \succ 0 \Leftrightarrow A \in \mathbf{S}^n_{++}$
- sometimes, use Python script-like notations (with serious abuse of mathematical notations)
  - use $f : \mathbf{R} \to \mathbf{R}$ as if it were $f : \mathbf{R}^n \to \mathbf{R}^n$, *e.g.*,

    $$\exp(x) = (\exp(x_1), \ldots, \exp(x_n)) \quad \text{for } x \in \mathbf{R}^n$$

    and

    $$\log(x) = (\log(x_1), \ldots, \log(x_n)) \quad \text{for } x \in \mathbf{R}^n_{++}$$

    which corresponds to Python code `numpy.exp(x)` or `numpy.log(x)` where `x` is instance of `numpy.ndarray`, *i.e.*, `numpy` array
  - use $\sum x$ to mean $\mathbf{1}^T x$ for $x \in \mathbf{R}^n$, *i.e.*

    $$\sum x = x_1 + \cdots + x_n$$

    which corresponds to Python code `x.sum()` where `x` is `numpy` array

– use x/y for $x, y \in \mathbf{R}^n$ to mean

$$\left[\begin{array}{ccc} x_1/y_1 & \cdots & x_n/y_n \end{array}\right]^T$$

which corresponds to Python code `x / y` where `x` and `y` are 1-d `numpy` arrays

– use X/Y for $X, Y \in \mathbf{R}^{m\times n}$ to mean

$$\left[\begin{array}{cccc} X_{1,1}/Y_{1,1} & X_{1,2}/Y_{1,2} & \cdots & X_{1,n}/Y_{1,n} \\ X_{2,1}/Y_{2,1} & X_{2,2}/Y_{2,2} & \cdots & X_{2,n}/Y_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{m,1}/Y_{m,1} & X_{m,2}/Y_{m,2} & \cdots & X_{m,n}/Y_{m,n} \end{array}\right]$$

which corresponds to Python code `X / Y` where `X` and `Y` are 2-d `numpy` arrays

Some definitions

Definition 1. [infinitely often - i.o.] *statement P_n, said to happen* infinitely often *or* i.o. *if*

$$(\forall N \in \mathbf{N})\,(\exists n > N)\,(P_n)$$

Definition 2. [almost everywhere - a.e.] *statement $P(x)$, said to happen* almost everywhere *or* a.e. *or* almost surely *or* a.s. *(depending on context) associated with measure space $(X, \mathscr{B}, \mu)$ if*

$$\mu\{x|P(x)\} = 1$$

or equivalently

$$\mu\{x| \sim P(x)\} = 0$$

Some conventions

- (for some subjects) use following conventions
  - $0 \cdot \infty = \infty \cdot 0 = 0$
  - $(\forall x \in \mathbf{R}_{++})(x \cdot \infty = \infty \cdot x = \infty)$
  - $\infty \cdot \infty = \infty$

Real Analysis

Set Theory

Some principles

Principle 1. [principle of mathematical induction]

$$P(1)\&[P(n \Rightarrow P(n+1)] \Rightarrow (\forall n \in \mathbf{N})P(n)$$

Principle 2. [well ordering principle] *each nonempty subset of* $\mathbf{N}$ *has a smallest element*

Principle 3. [principle of recursive definition] *for* $f : X \to X$ *and* $a \in X$*, exists unique infinite sequence* $\langle x_n \rangle_{n=1}^{\infty} \subset X$ *such that*

$$x_1 = a$$

and

$$(\forall n \in \mathbf{N})\,(x_{n+1} = f(x_n))$$

- note that Principle 1 $\Leftrightarrow$ Principle 2 $\Rightarrow$ Principle 3

Some definitions for functions

Definition 3. [functions] *for* $f : X \to Y$

- *terms,* map *and* function*, exterchangeably used*
- X *and* Y*, called* domain of f *and* codomain of f *respectively*
- $\{f(x)|x \in X\}$*, called* range of f
- *for* $Z \subset Y$*,* $f^{-1}(Z) = \{x \in X|f(x) \in Z\} \subset X$*, called* preimage *or* inverse image of Z under f
- *for* $y \in Y$*,* $f^{-1}(\{y\})$*, called* fiber of f over y
- f*, called* injective *or* injection *or* one-to-one *if* $(\forall x \neq v \in X) \, (f(x) \neq f(v))$
- f*, called* surjective *or* surjection *or* onto *if* $(\forall x \in X) \, (\exists y in Y) \, (y = f(x))$
- f*, called* bijective *or* bijection *if* f *is both injective and surjective, in which case,* X *and* Y*, said to be* one-to-one correspondece *or* bijective correspondece
- $g : Y \to X$*, called* left inverse *if* $g \circ f$ *is identity function*
- $h : Y \to X$*, called* right inverse *if* $f \circ h$ *is identity function*

Some properties of functions

Lemma 1. [functions] *for* $f : X \to Y$

- f *is injective* if and only if f *has left inverse*

- f *is surjective* if and only if f *has right inverse*

- *hence,* f *is bijective* if and only if f *has both left and right inverse because if* g *and* h *are left and right inverses respectively,* $g = g \circ (f \circ h) = (g \circ f) \circ h = h$

- *if* $|X| = |Y| < \infty$, f *is injective* if and only if f *is surjective* if and only if f *is bijective*

Countability of sets

- set A is countable if range of some function whose domain is **N**

- **N**, **Z**, **Q**: countable

- **R**: *not* countable

Limit sets

- for sequence, $\langle A_n \rangle$, of subsets of X
  - *limit superior or limsup of* $\langle A_n \rangle$, defined by

$$\limsup \langle A_n \rangle = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$$

  - *limit inferior or liminf of* $\langle A_n \rangle$, defined by

$$\liminf \langle A_n \rangle = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m$$

- always

$$\liminf \langle A_n \rangle \subset \limsup \langle A_n \rangle$$

- when $\liminf \langle A_n \rangle = \limsup \langle A_n \rangle$, sequence, $\langle A_n \rangle$, said to *converge to it*, denote

$$\lim \langle A_n \rangle = \liminf \langle A_n \rangle = \limsup \langle A_n \rangle = A$$

Algebras of sets

- collection $\mathscr{A}$ of subsets of X called *algebra* or *Boolean algebra* if

$$(\forall A, B \in \mathscr{A})(A \cup B \in \mathscr{A}) \text{ and } (\forall A \in \mathscr{A})(\tilde{A} \in \mathscr{A})$$

  - $(\forall A_1, \ldots, A_n \in \mathscr{A})(\cup_{i=1}^n A_i \in \mathscr{A})$
  - $(\forall A_1, \ldots, A_n \in \mathscr{A})(\cap_{i=1}^n A_i \in \mathscr{A})$
- algebra $\mathscr{A}$ called *σ-algebra* or *Borel field* if
  - every union of a countable collection of sets in $\mathscr{A}$ is in $\mathscr{A}$, *i.e.*,

$$(\forall \langle A_i \rangle)(\cup_{i=1}^\infty A_i \in \mathscr{A})$$

- given sequence of sets in algebra $\mathscr{A}$, $\langle A_i \rangle$, exists disjoint sequence, $\langle B_i \rangle$ such that

$$B_i \subset A_i \text{ and } \bigcup_{i=1}^\infty B_i = \bigcup_{i=1}^\infty A_i$$

Algebras generated by subsets

- *algebra generated by* collection of subsets of X, $\mathcal{C}$, can be found by

$$\mathscr{A} = \bigcap \{\mathscr{B} | \mathscr{B} \in \mathcal{F}\}$$

  where $\mathcal{F}$ is family of all algebras containing $\mathcal{C}$
  – smallest algebra $\mathscr{A}$ containing $\mathcal{C}$, $i.e.$,

$$(\forall \mathscr{B} \in \mathcal{F})(\mathscr{A} \subset \mathscr{B})$$

- *σ-algebra generated by* collection of subsets of X, $\mathcal{C}$, can be found by

$$\mathscr{A} = \bigcap \{\mathscr{B} | \mathscr{B} \in \mathcal{G}\}$$

  where $\mathcal{G}$ is family of all σ-algebras containing $\mathcal{C}$
  – smallest σ-algebra $\mathscr{A}$ containing $\mathcal{C}$, $i.e.$,

$$(\forall \mathscr{B} \in \mathcal{G})(\mathscr{A} \subset \mathscr{B})$$

Relation

- x said to *stand in relation* $\mathbf{R}$ to y, denoted by $x \ \mathbf{R} \ y$
- $\mathbf{R}$ said to *be relation on* X if $x \ \mathbf{R} \ y \Rightarrow x \in X$ and $y \in X$
- $\mathbf{R}$ is
  - transitive if $x \ \mathbf{R} \ y$ and $y \ \mathbf{R} \ z \Rightarrow x \ \mathbf{R} \ z$
  - symmetric if $x \ \mathbf{R} \ y = y \ \mathbf{R} \ x$
  - reflexive if $x \ \mathbf{R} \ x$
  - antisymmetric if $x \ \mathbf{R} \ y$ and $y \ \mathbf{R} \ x \Rightarrow x = y$
- $\mathbf{R}$ is
  - *equivalence relation* if transitive, symmetric, and reflexive, $e.g.$, modulo
  - *partial ordering* if transitive and antisymmetric, $e.g.$, "$\subset$"
  - *linear (or simple) ordering* if transitive, antisymmetric, and $x \ \mathbf{R} \ y$ or $y \ \mathbf{R} \ x$ for all $x, y \in X$
    - $e.g.$, "$\geq$" linearly orders **R** while "$\subset$" does not $\mathcal{P}(X)$

Ordering

- given partial order, $\prec$, a is
  - a first/smallest/least element if $x \neq a \Rightarrow a \prec x$
  - a last/largest/greatest element if $x \neq a \Rightarrow x \prec a$
  - a minimal element if $x \neq a \Rightarrow x \not\prec a$
  - a maximal element if $x \neq a \Rightarrow a \not\prec x$
- partial ordering $\prec$ is
  - strict partial ordering if $x \not\prec x$
  - reflexive partial ordering if $x \prec x$
- strict linear ordering $<$ is
  - *well ordering* for X if every nonempty set contains a first element

Axiom of choice and equivalent principles

Axiom 1. [axiom of choice] *given a collection of nonempty sets, $\mathcal{C}$, there exists $f : \mathcal{C} \to \cup_{A \in \mathcal{C}} A$ such that*

$$(\forall A \in \mathcal{C}\ ) \left(f(A) \in A \right)$$

- also called *multiplicative axiom* - preferred to be called to axiom of choice by Bertrand Russell for reason writte on page 20
- no problem when $\mathcal{C}$ is finite
- need axiom of choice when $\mathcal{C}$ is not finite

Principle 4. [Hausdorff maximal principle] *for particial ordering $\prec$ on X, exists a maximal linearly ordered subset $S \subset X$, $i.e.$, S is linearity ordered by $\prec$ and if $S \subset T \subset X$ and T is linearly ordered by $\prec$, $S = T$*

Principle 5. [well-ordering principle] *every set X can be well ordered, $i.e.$, there is a relation $<$ that well orders X*

- note that Axiom 1 $\Leftrightarrow$ Principle 4 $\Leftrightarrow$ Principle 5

Infinite direct product

Definition 4. [direct product] *for collection of sets,* $\langle X_\lambda \rangle$, *with index set,* Λ,

$$\bigtimes_{\lambda \in \Lambda} X_\lambda$$

called direct product

- *for* $z = \langle x_\lambda \rangle \in \bigtimes X_\lambda$, x_λ *called* λ-th coordinate of z

- if one of X_λ is empty, $\bigtimes X_\lambda$ is empty
- *axiom of choice* is equivalent to converse, $i.e.$, if none of X_λ is empty, $\bigtimes X_\lambda$ is not empty
  if one of X_λ is empty, $\bigtimes X_\lambda$ is empty
- this is why Bertrand Russell prefers *multiplicative axiom* to *axiom of choice* for name of axiom (Axiom 1)

Real Number System

Field axioms

- field axioms - for every $x, y, z \in \mathbf{F}$
  - $(x + y) + z = x + (y + z)$ - additive associativity
  - $(\exists 0 \in \mathbf{F})(\forall x \in \mathbf{F})(x + 0 = x)$ - additive identity
  - $(\forall x \in \mathbf{F})(\exists w \in \mathbf{F})(x + w = 0)$ - additive inverse
  - $x + y = y + x$ - additive commutativity
  - $(xy)z = x(yz)$ - multiplicative associativity
  - $(\exists 1 \neq 0 \in \mathbf{F})(\forall x \in \mathbf{F})(x \cdot 1 = x)$ - multiplicative identity
  - $(\forall x \neq 0 \in \mathbf{F})(\exists w \in \mathbf{F})(xw = 1)$ - multiplicative inverse
  - $x(y + z) = xy + xz$ - distributivity
  - $xy = yx$ - multiplicative commutativity
- system (set with $+$ and $\cdot$) satisfying axiom of field called *field*
  - $e.g.$, field of module p where p is prime, $\mathbf{F}_p$

Axioms of order

- axioms of order - subset, $\mathbf{F}_{++} \subset \mathbf{F}$, of positive (real) numbers satisfies
  - $x, y \in \mathbf{F}_{++} \Rightarrow x + y \in \mathbf{F}_{++}$
  - $x, y \in \mathbf{F}_{++} \Rightarrow xy \in \mathbf{F}_{++}$
  - $x \in \mathbf{F}_{++} \Rightarrow -x \not\in \mathbf{F}_{++}$
  - $x \in \mathbf{F} \Rightarrow x = 0 \vee x \in \mathbf{F}_{++} \vee -x \in \mathbf{F}_{++}$

- system satisfying field axioms & axioms of order called *ordered field*
  - $e.g.$, set of real numbers ($\mathbf{R}$), set of rational numbers ($\mathbf{Q}$)

Axiom of completeness

- completeness axiom
  - every nonempty set S of real numbers which has an upper bound has a least upper bound, $i.e.$,
  $$\{l|(\forall x \in S)(l \leq x)\}$$
  has least element.
  - use $\inf S$ and $\sup S$ for least and greatest element (when exist)
- ordered field that is complete is *complete ordered field*
  - $e.g.$, **R** (with $+$ and $\cdot$)

$\Rightarrow$ axiom of Archimedes

- given any $x \in$ **R**, there is an integer n such that $x < n$

$\Rightarrow$ corollary

- given any $x < y \in$ **R**, exists $r \in$ **Q** such tat $x < r < y$

Sequences of R

- sequence of **R** denoted by $\langle x_i \rangle_{i=1}^{\infty}$ or $\langle x_i \rangle$
  - mapping from **N** to **R**
- limit of $\langle x_n \rangle$ denoted by $\lim_{n\to\infty} x_n$ or $\lim x_n$ - defined by $a \in \mathbf{R}$

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N})(n \geq N \Rightarrow |x_n - a| < \epsilon)$$

  - $\lim x_n$ unique if exists
- $\langle x_n \rangle$ called Cauchy sequence if

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N})(n, m \geq N \Rightarrow |x_n - x_m| < \epsilon)$$

- Cauchy criterion - characterizing complete metric space (including **R**)
  - sequence converges *if and only if* Cauchy sequence

Other limits

- cluster point of $\langle x_n \rangle$ - defined by $c \in \mathbf{R}$

$$(\forall \epsilon > 0, N \in \mathbf{N})(\exists n > N)(|x_n - c| < \epsilon)$$

- limit superior or limsup of $\langle x_n \rangle$

$$\limsup x_n = \inf_n \sup_{k>n} x_k$$

- limit inferior or liminf of $\langle x_n \rangle$

$$\liminf x_n = \sup_n \inf_{k>n} x_k$$

- $\liminf x_n \leq \limsup x_n$
- $\langle x_n \rangle$ converges *if and only if* $\liminf x_n = \limsup x_n$ $(=\lim x_n)$

Open and closed sets

- O called *open* if

$$(\forall x \in O)(\exists \delta > 0)(y \in \mathbf{R})(|y - x| < \delta \Rightarrow y \in O)$$

  - intersection of finite collection of open sets is open
  - union of any collection of open sets is open

- $\overline{E}$ called *closure* of E if

$$(\forall x \in \overline{E} \;\&\; \delta > 0)(\exists y \in E)(|x - y| < \delta)$$

- F called *closed* if

$$F = \overline{F}$$

  - union of finite collection of closed sets is closed
  - intersection of any collection of closed sets is closed

Open and closed sets - facts

- *every open set is union of countable collection of disjoint open intervals*

- (Lindelöf) any collection $\mathcal{C}$ of open sets has a countable subcollection $\langle O_i \rangle$ such that

$$\bigcup_{O \in \mathcal{C}} O = \bigcup_i O_i$$

  – equivalently, any collection $\mathcal{F}$ of closed sets has a countable subcollection $\langle F_i \rangle$ such that

$$\bigcap_{O \in \mathcal{F}} F = \bigcap_i F_i$$

Covering and Heine-Borel theorem

- collection $\mathcal{C}$ of sets called *covering* of A if

$$A \subset \bigcup_{O \in \mathcal{C}} O$$

  - $\mathcal{C}$ said to *cover* A
  - $\mathcal{C}$ called *open covering* if every $O \in \mathcal{C}$ is open
  - $\mathcal{C}$ called *finite covering* if $\mathcal{C}$ is finite

- *Heine-Borel theorem -* for any closed and bounded set, every open covering has finite subcovering

- corollary
  - any collection $\mathcal{C}$ of closed sets including at least one bounded set every finite subcollection of which has nonempty intersection has nonempty intersection.

Continuous functions

- f (with domain D) called *continuous at* x if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall y \in D)(|y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon)$$

- f called *continuous on* $A \subset D$ if f is continuous at every point in A

- f called *uniformly continuous on* $A \subset D$ if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x, y \in D)(|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon)$$

Continuous functions - facts

- f is continuous *if and only if* for every open set O (in co-domain), $f^{-1}(O)$ is open
- f continuous on closed and bounded set is uniformly continuous
- *extreme value theorem* - f continuous on closed and bounded set, F, is *bounded on F and assumes its maximum and minimum on F*

$$(\exists x_1, x_2 \in F)(\forall x \in F)(f(x_1) \leq f(x) \leq f(x_2))$$

- *intermediate value theorem* - for f continuous on $[a, b]$ with $f(a) \leq f(b)$,

$$(\forall d)(f(a) \leq d \leq f(b))(\exists c \in [a, b])(f(c) = d)$$

Borel sets and Borel σ-algebra

- *Borel set*
  - any set that can be formed from open sets (or, equivalently, from closed sets) through the operations of countable union, countable intersection, and relative complement

- *Borel algebra* or *Borel σ-algebra*
  - *smallest σ-algebra containing all open sets*

  - also
    - smallest σ-algebra containing all closed sets
    - smallest σ-algebra containing all open intervals (due to statement on page 28)

Various Borel sets

- countable union of closed sets (in **R**), called *an* F_σ (F for closed & σ for sum)
  - thus, every countable set, every closed set, every open interval, every open sets, is an F_σ (note $(a,b) = \bigcup_{n=1}^{\infty}[a+1/n, b-1/n]$)
  - countable union of sets in F_σ again is an F_σ
- countable intersection of open sets called *a* G_δ (G for open & δ for durchschnitt - average in German)
  - complement of F_σ is a G_δ and vice versa
- F_σ and G_δ are simple types of Borel sets
- countable intersection of F_σ's is $F_{\sigma\delta}$, countable union of $F_{\sigma\delta}$'s is $F_{\sigma\delta\sigma}$, countable intersection of $F_{\sigma\delta\sigma}$'s is $F_{\sigma\delta\sigma\delta}$, $etc.$, & likewise for $G_{\delta\sigma\dots}$
- below are all classes of Borel sets, but not every Borel set belongs to one of these classes

$$F_\sigma, F_{\sigma\delta}, F_{\sigma\delta\sigma}, F_{\sigma\delta\sigma\delta}, \dots, G_\delta, G_{\delta\sigma}, G_{\delta\sigma\delta}, G_{\delta\sigma\delta\sigma}, \dots,$$

Measure and Integration

Purpose of integration theory

- purpose of "measure and integration" slides
  - abstract (out) most important properties of Lebesgue measure and Lebesgue integration

- provide certain *axioms that Lebesgue measure satisfies*

- base our integration theory on these axioms

- hence, our theory valid for every system satisfying the axioms

Measurable space, measure, and measure space

- family of subsets containing $\emptyset$ closed under countable union and completement, called *σ-algebra*
- mapping of sets to extended real numbers, called *set function*
- $(X, \mathscr{B})$ with set, X, and σ-algebra of X, $\mathscr{B}$, called *measurable space*
  - $A \in \mathscr{B}$, said to be *measurable (with respect to $\mathscr{B}$)*
- nonnegative set function, μ, defined on $\mathscr{B}$ satisfying $\mu(\emptyset) = 0$ and for every disjoint, $\langle E_n \rangle_{n=1}^{\infty} \subset \mathscr{B}$,

  $$\mu\left(\bigcup E_n\right) = \sum \mu E_n$$

  called *measure on* measurable space, $(X, \mathscr{B})$
- measurable space, $(X, \mathscr{B})$, equipped with measure, μ, called *measure space* and denoted by $(X, \mathscr{B}, \mu)$

Measure space examples

- $(\mathbf{R}, \mathcal{M}, \mu)$ with Lebesgue measurable sets, $\mathcal{M}$, and Lebesgue measure, μ
- $([0,1], \{A \in \mathcal{M} | A \subset [0,1]\}, \mu)$ with Lebesgue measurable sets, $\mathcal{M}$, and Lebesgue measure, μ
- $(\mathbf{R}, \mathscr{B}, \mu)$ with class of Borel sets, $\mathscr{B}$, and Lebesgue measure, μ
- $(\mathbf{R}, \mathcal{P}(\mathbf{R}), \mu_C)$ with set of all subsets of $\mathbf{R}$, $\mathcal{P}(\mathbf{R})$, and counting measure, μ_C
- interesting (and bizarre) example
  - $(X, \mathcal{A}, \mu_B)$ with any uncountable set, X, family of either countable or complement of countable set, $\mathcal{A}$, and measure, μ_B, such that $\mu_B A = 0$ for countable $A \subset X$ and $\mu_B B = 1$ for uncountable $B \subset X$

More properties of measures

- for $A, B \in \mathscr{B}$ with $A \subset B$

$$\mu A \leq \mu B$$

- for $\langle E_n \rangle \subset \mathscr{B}$ with $\mu E_1 < \infty$ and $E_{n+1} \subset E_n$

$$\mu \left(\bigcap E_n \right) = \lim \mu E_n$$

- for $\langle E_n \rangle \subset \mathscr{B}$

$$\mu \left(\bigcup E_n \right) \leq \sum \mu E_n$$

Finite and σ-finite measures

- measure, μ, with $\mu(X) < \infty$, called *finite*
- measure, μ, with $X = \bigcup X_n$ for some $\langle X_n \rangle$ and $\mu(X_n) < \infty$, called *σ-finite*
  - always can take $\langle X_n \rangle$ with disjoint X_n
- Lebesgue measure on $[0, 1]$ is finite
- Lebesgue measure on **R** is σ-finite
- countering measure on uncountable set is *not* σ-measure

Sets of finite and σ-finite measure

- set, $E \in \mathscr{B}$, with $\mu E < \infty$, said to be *of finite measure*
- set that is countable union of measurable sets of finite measure, said to be *of σ-finite measure*
- measurable set contained in set of σ-finite measure, is of σ-finite measure
- countable union of sets of σ-finite measure, is of σ-finite measure
- when μ is σ-finite, every measurable set is of σ-finite

Semifinite measures

- roughly speacking, nearly all familiar properties of Lebesgue measure and Lebesgue integration hold for arbitrary σ-finite measure
- many treatment of abstract measure theory limit themselves to σ-finite measures

- many parts of general theory, however, do *not* required assumption of σ-finiteness
- undesirable to have development unnecessarily restrictive

- measure, μ, for which every measurable set of infinite measure contains measurable sets of arbitrarily large finite measure, said to be *semifinite*
- every σ-finite measure is semifinite measure while measure, μ_B, on page 37 is not

Complete measure spaces

- measure space, $(X, \mathscr{B}, \mu)$, for which $\mathscr{B}$ contains all subsets of sets of measure zero, said to be *complete*, $i.e.$,

$$(\forall B \in \mathscr{B} \text{ with } \mu B = 0)(A \subset B \Rightarrow A \in \mathscr{B})$$

  - $e.g.$, Lebesgue measure is complete, but Lebesgue measure restricted to σ-algebra of Borel sets is *not*
- every measure space can be *completed* by addition of subsets of sets of measure zero
- for $(X, \mathscr{B}, \mu)$, can find *complete* measure space $(X, \mathscr{B}_0, \mu_0)$ such that

  - $\mathscr{B} \subset \mathscr{B}_0$
  - $E \in \mathscr{B} \Rightarrow \mu E = \mu_0 E$
  - $E \in \mathscr{B}_0 \Leftrightarrow E = A \cup B$ where $B, C \in \mathscr{B}, \mu C = 0, A \subset C$

  - $(X, \mathscr{B}_0, \mu_0)$ called *completion* of $(X, \mathscr{B}, \mu)$

Local measurability and saturatedness

- for $(X, \mathscr{B}, \mu)$, $E \subset X$ for which $(\forall B \in \mathscr{B}$ with $\mu B < \infty)(E \cap B \in \mathscr{B})$, said to be *locally measurable*
- collection, $\mathscr{C}$, of all locally measurable sets is σ-algebra containing $\mathscr{B}$
- measure for which every locally measurable set is measurable, said to be *saturated*
- every σ-finite measure is saturated
- measure can be extended to saturated measure, but (unlike completion) extension is not unique
  - can take $\mathscr{C}$ as extension for locally measurable sets, but measure can be extended on $\mathscr{C}$ in more than one ways

Measurable functions

- concept and properties of measurable functions in abstract measurable space almost identical with those of Lebesgue measurable functions (page **??**)
- theorems and facts are essentially same as those of Lebesgue measurable functions
- assume measurable space, $(X, \mathscr{B})$
- for $f : X \to \mathbf{R} \cup \{-\infty, \infty\}$, following are equivalent
  - $(\forall a \in \mathbf{R})(\{x \in X | f(x) < a\} \in \mathscr{B})$
  - $(\forall a \in \mathbf{R})(\{x \in X | f(x) \leq a\} \in \mathscr{B})$
  - $(\forall a \in \mathbf{R})(\{x \in X | f(x) > a\} \in \mathscr{B})$
  - $(\forall a \in \mathbf{R})(\{x \in X | f(x) \geq a\} \in \mathscr{B})$
- $f : X \to \mathbf{R} \cup \{-\infty, \infty\}$ for which any one of above four statements holds, called *measurable* or *measurable with respect to* $\mathscr{B}$

(refer to page **??** for Lebesgue counterpart)

Properties of measurable functions

- **Theorem 1. [measurability preserving function operations]** *for measurable functions, f and g, and $c \in \mathbf{R}$*
  - $f + c$, cf, $f + g$, fg, $f \vee g$ *are measurable*

- **Theorem 2. [limits of measurable functions]** *for every measurable function sequence,* $\langle f_n \rangle$
  - $\sup f_n$, $\limsup f_n$, $\inf f_n$, $\liminf f_n$ *are measurable*
  - *thus,* $\lim f_n$ *is measurable if exists*

(refer to page **??** for Lebesgue counterpart)

Simple functions and other properties

- φ called *simple function* if for distinct $\langle c_i \rangle_{i=1}^n$ and measurable sets, $\langle E_i \rangle_{i=1}^n$

$$\varphi(x) = \sum_{i=1}^n c_i \chi_{E_i}(x)$$

(refer to page **??** for Lebesgue counterpart)

- for nonnegative measurable function, f, exists nondecreasing sequence of simple functions, $\langle \varphi_n \rangle$, *i.e.*, $\varphi_{n+1} \geq \varphi_n$ such that for every point in X

$$f = \lim \varphi_n$$

  - for f defined on σ-finite measure space, we may choose $\langle \varphi_n \rangle$ so that every φ_n vanishes outside set of finite measure
- for complete measure, μ, f measurable and $f = g$ a.e. imply measurability of g

Define measurable function by ordinate sets

- $\{x|f(x) < \alpha\}$ sometimes called *ordinate sets*, which is nondecreasing in α
- below says when given nondecreasing ordinate sets, we can find f satisfying

$$\{x|f(x) < \alpha\} \subset B_\alpha \subset \{x|f(x) \leq \alpha\}$$

- for nondecreasing function, $h : D \to \mathscr{B}$, for dense set of real numbers, D, *i.e.*, $B_\alpha \subset B_\beta$ for all $\alpha < \beta$ where $B_\alpha = h(\alpha)$, exists unique measurable function, $f : X \to \mathbf{R} \cup \{-\infty, \infty\}$ such that $f \leq \alpha$ on B_α and $f \geq \alpha$ on $X \sim B_\alpha$
- can relax some conditions and make it a.e. version as below
- for function, $h : D \to \mathscr{B}$, for dense set of real numbers, D, such that $\mu(B_\alpha \sim B_\beta) = 0$ for all $\alpha < \beta$ where $B_\alpha = h(\alpha)$, exists measurable function, $f : X \to \mathbf{R} \cup \{-\infty, \infty\}$ such that $f \leq \alpha$ a.e. on B_α and $f \geq \alpha$ a.e. on $X \sim B_\alpha$
  - if g has the same property, $f = g$ a.e.

Integration

- many definitions and proofs of Lebesgue integral depend only on properties of Lebesgue measure which are also true for arbitrary measure in abstract measure space (page **??**)
- integral of nonnegative simple function, $\varphi(x) = \sum_{i=1}^n c_i \chi_{E_i}(x)$, on measurable set, E, defined by

$$\int_E \varphi d\mu = \sum_{i=1}^n c_i \mu(E_i \cap E)$$

  – independent of representation of φ

(refer to page **??** for Lebesgue counterpart)

- for $a, b \in \mathbf{R}_{++}$ and nonnegative simple functions, φ and ψ

$$\int (a\varphi + b\psi) = a \int \varphi + b \int \psi$$

(refer to page **??** for Lebesgue counterpart)

Integral of bounded functions

- for bounded function, f, identically zero outside measurable set of finite measure

$$\sup_{\varphi:\ \mathrm{simple},\ \varphi\leq f} \int \varphi = \inf_{\psi:\ \mathrm{simple},\ f\leq\psi} \int \psi$$

  if and only if $f = g$ a.e. for measurable function, g

(refer to page **??** for Lebesgue counterpart)

- but, *$f = g$ a.e. for measurable function, g,* if and only if *f is measurable with respect to completion of μ, $\bar{\mu}$*
- *natural class of functions to consider for integration theory are those measurable with respect to completion of μ*
- thus, shall either assume μ is complete measure or define integral with respect to μ to be integral with respect to completion of μ depending on context unless otherwise specified

Difficulty of general integral of nonnegative functions

- for Lebesgue integral of nonnegative functions (page **??**)
  - first define integral for bounded measurable functions
  - define integral of nonnegative function, f as supremum of integrals of all bounded measurable functions, $h \leq f$, vanishing outside measurable set of finite measure

- unfortunately, not work in case that measure is not semifinite
  - $e.g.$, if $\mathscr{B} = \{\emptyset, X\}$ with $\mu\emptyset = 0$ and $\mu X = \infty$, we want $\int 1 d\mu = \infty$, but only bounded measurable function vanishing outside measurable set of finite measure is $h \equiv 0$, hence, $\int g d\mu = 0$

- to avoid this difficulty, we define integral of nonnegative measurable function directly in terms of integrals of nonnegative simple functions

Integral of nonnegative functions

- for measurable function, $f : X \to \mathbf{R} \cup \{\infty\}$, on measure space, $(X, \mathscr{B}, \mu)$, define *integral of nonnegative extended real-valued measurable function*

$$\int f d\mu = \sup_{\varphi:\ \text{simple function},\ 0 \leq \varphi \leq f} \int \varphi d\mu$$

(refer to page **??** for Lebesgue counterpart)

- however, *definition of integral of nonnegative extended real-valued measurable function* can be awkward to apply because
  - taking supremum over large collection of simple functions
  - *not clear from definition that* $\int (f+g) = \int f + \int g$
- thus, first establish some convergence theorems, and determine value of $\int f$ as limit of $\int \varphi_n$ for increasing sequence, $\langle \varphi_n \rangle$, of simple functions converging to f

Fatou's lemma and monotone convergence theorem

- *Fatou's lemma* - for nonnegative measurable function sequence, $\langle f_n \rangle$, with $\lim f_n = f$ a.e. on measurable set, E
$$\int_E f \leq \liminf \int_E f_n$$

- *monotone convergence theorem* - for nonnegative measurable function sequence, $\langle f_n \rangle$, with $f_n \leq f$ for all n and with $\lim f_n = f$ a.e.
$$\int_E f = \lim \int_E f_n$$

(refer to page **??** for Lebesgue counterpart)

Integrability of nonnegative functions

- for nonnegative measurable functions, f and g, and $a, b \in \mathbf{R}_+$

$$\int (af + bg) = a \int f + b \int g \;\&\; \int f \geq 0$$

  - equality holds *if and only if* $f = 0$ a.e.

(refer to page **??** for Lebesgue counterpart)

- monotone convergence theorem together with above yields, for nonnegative measurable function sequence, $\langle f_n \rangle$

$$\int \sum f_n = \sum \int f_n$$

- measurable nonnegative function, f, with

$$\int_E f d\mu < \infty$$

  said to be *integral (over measurable set, E, with respect to μ)*

(refer to page **??** for Lebesgue counterpart)

Integral

- arbitrary function, f, for which both f^+ and f^- are integrable, said to be *integrable*

- in this case, define *integral*

$$\int_E f = \int_E f^+ - \int_E f^-$$

(refer to page **??** for Lebesgue counterpart)

Properties of integral

- for f and g integrable on measure set, E, and $a, b \in \mathbf{R}$
  - $af + bg$ is integral and
$$\int_E (af + bg) = a \int_E f + b \int_E g$$
  - if $|h| \leq |f|$ and h is measurable, then h is integrable
  - if $f \geq g$ a.e.
$$\int f \geq \int g$$

(refer to page **??** for Lebesgue counterpart)

Lebesgue convergence theorem

- *Lebesgue convergence theorem -* for integral, g, over E and sequence of measurable functions, $\langle f_n \rangle$, with $\lim f_n(x) = f(x)$ a.e. on E, if

$$|f_n(x)| \leq g(x)$$

then

$$\int_E f = \lim \int_E f_n$$

(refer to page **??** for Lebesgue counterpart)

Setwise convergence of sequence of measures

- preceding convergence theorems assume fixed measure, μ

- can generalize by allowing measure to vary

- given measurable space, $(X, \mathscr{B})$, sequence of set functions, $\langle \mu_n \rangle$, defined on $\mathscr{B}$, satisfying
$$(\forall E \in \mathscr{B})(\lim \mu_n E = \mu E)$$
for some set function, μ, defined on $\mathscr{B}$, said to *converge setwise* to μ

General convergence theorems

- *generalization of Fatou's leamma* - for measurable space, $(X, \mathscr{B})$, sequence of measures, $\langle \mu_n \rangle$, defined on $\mathscr{B}$, converging setwise to μ, defined on $\mathscr{B}$, and sequence of nonnegative functions, $\langle f_n \rangle$, each measurable with respect to μ_n, converging pointwise to function, f, measurable with respect to μ (compare with Fatou's lemma on page 52)

$$\int f d\mu \leq \liminf \int f_n d\mu_n$$

- *generalization of Lebesgue convergence theorem* - for measurable space, $(X, \mathscr{B})$, sequence of measures, $\langle \mu_n \rangle$, defined on $\mathscr{B}$, converging setwise to μ, defined on $\mathscr{B}$, and sequences of functions, $\langle f_n \rangle$ and $\langle g_n \rangle$, each of f_n and g_n, measurable with respect to μ_n, converging pointwise to f and g, measurable with respect to μ, respectively, such that (compare with Lebesgue convergence theorem on page 56)

$$\lim \int g_n d\mu_n = \int g d\mu < \infty$$

satisfy

$$\lim \int f_n d\mu_n = \int f \mu$$

L^p spaces

- for complete measure space, $(X, \mathscr{B}, \mu)$
  - space of measurable functions on X with with $\int |f|^p < \infty$, for which element equivalence is defined by being equal a.e., called *L^p spaces* denoted by $L^p(\mu)$
  - space of bounded measure functions, called *L^∞ space* denoted by $L^\infty(\mu)$

- norms
  - for $p \in [1, \infty)$

$$\|f\|_p = \left(\int |f|^p d\mu \right)^{1/p}$$

  - for $p = \infty$

$$\|f\|_\infty = \mathrm{ess\ sup}|f| = \inf \{ |g(x)|| \text{ measurable } g \text{ with } g = f \text{ a.e.} \}$$

- for $p \in [1, \infty]$, spaces, $L^p(\mu)$, are Banach spaces

Hölder's inequality and Littlewood's second principle

- *Hölder's inequality* - for $p, q \in [1, \infty]$ with $1/p + 1/q = 1$, $f \in L^p(\mu)$ and $g \in L^q(\mu)$ satisfy $fg \in L^1(\mu)$ and

$$\|fg\|_1 = \int |fg| d\mu \leq \|f\|_p \|g\|_q$$

(refer to page **??** for normed spaces counterpart)

- *complete measure space version of Littlewood's second principle* - for $p \in [1, \infty)$

$$(\forall f \in L^p(\mu), \epsilon > 0)$$

$$(\exists \text{ simple function } \varphi \text{ vanishing outside set of finite measure})$$

$$(\|f - \varphi\|_p < \epsilon)$$

(refer to page **??** for normed spaces counterpart)

Riesz representation theorem

- *Riesz representation theorem* - for $p \in [1, \infty)$ and bounded linear functional, F, on $L^p(\mu)$ and σ-finite measure, μ, exists *unique* $g \in L^q(\mu)$ where $1/p + 1/q = 1$ such that
$$F(f) = \int fg d\mu$$
where $\|F\| = \|g\|_q$

(refer to page **??** for normed spaces counterpart)

- if $p \in (1, \infty)$, Riesz representation theorem holds without assumption of σ-finiteness of measure

Measure and Outer Measure

General measures

- consider some ways of defining measures on σ-algebra

- recall that for Lebesgue measure
  - define measure for open intervals
  - define outer measure
  - define notion of measurable sets
  - finally derive Lebesgue measure

- one can do similar things in general, $e.g.$,
  - derive measure from outer measure
  - derive outer measure from measure defined on algebra of sets

Outer measure

- set function, $\mu^* : \mathcal{P}(X) \to [0, \infty]$, for space X, having following properties, called *outer measure*
  - $\mu^* \emptyset = 0$
  - $A \subset B \Rightarrow \mu^* A \leq \mu^* B$ (monotonicity)
  - $E \subset \bigcup_{n=1}^{\infty} E_n \Rightarrow \mu^* E \leq \sum_{n=1}^{\infty} \mu^* E_n$ (countable subadditivity)
- μ^* with $\mu^* X < \infty$ called *finite*
- set $E \subset X$ satisfying following property, said to be *measurable with respect to μ^**

$$(\forall A \subset X)(\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap \tilde{E}))$$

- class, $\mathscr{B}$, of μ^*-measurable sets is σ-algebra
- restriction of μ^* to $\mathscr{B}$ is complete measure on $\mathscr{B}$

Extension to measure from measure on an algebra

- set function, $\mu : \mathscr{A} \to [0, \infty]$, defined on algebra, $\mathscr{A}$, having following properties, called *measure on an algebra*
  - $\mu(\emptyset) = 0$
  - $(\forall$ disjoint $\langle A_n \rangle \subset \mathscr{A}$ with $\bigcup A_n \in \mathscr{A}) \left(\mu\left(\bigcup A_n\right) = \sum \mu A_n\right)$

- *measure on an algebra,* $\mathscr{A}$, is measure *if and only if* $\mathscr{A}$ is σ-algebra

- can extend measure on an algebra to measure defined on σ-algebra, $\mathscr{B}$, containing $\mathscr{A}$, by
  - constructing outer measure μ^* from μ
  - deriving desired extension $\bar{\mu}$ induced by μ^*

– process by which constructing μ^* from μ similar to constructing Lebesgue outer measure from lengths of intervals

Outer measure constructed from measure on an algebra

— given measure, μ, on an algebra, $\mathscr{A}$

- define set function, $\mu^* : \mathcal{P}(X) \to [0, \infty]$, by

$$\mu^* E = \inf_{\langle A_n \rangle \subset \mathscr{A},\ E \subset \bigcup A_n} \sum \mu A_n$$

- μ^* called *outer measure induced by μ*

— then

- for $A \in \mathscr{A}$ and $\langle A_n \rangle \subset \mathscr{A}$ with $A \subset \bigcup A_n$, $\mu A \leq \sum \mu A_n$
- hence, $(\forall A \in \mathscr{A})(\mu^* A = \mu A)$
- μ^* is outer measure
- every $A \in \mathscr{A}$ is measurable with respect to μ^*

Regular outer measure

- for algebra, $\mathscr{A}$
  - $\mathscr{A}_\sigma$ denote sets that are countable unions of sets of $\mathscr{A}$
  - $\mathscr{A}_{\sigma\delta}$ denote sets that are countable intersections of sets of $\mathscr{A}_\sigma$
- given measure, μ, on an algebra, $\mathscr{A}$ and outer measure, μ^* induced by μ, for every $E \subset X$ and every $\epsilon > 0$, exists $A \in \mathscr{A}_\sigma$ and $B \in \mathscr{A}_{\sigma\delta}$ with $E \subset A$ and $E \subset B$

$$\mu^* A \leq \mu^* E + \epsilon \text{ and } \mu^* E = \mu^* B$$

- outer measure, μ^*, with below property, said to be *regular*

$$(\forall E \subset X, \epsilon > 0)(\exists\ \mu^*\text{-measurable set } A \text{ with } E \subset A)(\mu^* A \subset \mu^* E + \epsilon)$$

- every outer measure induced by measure on an algebra is regular outer measure

Carathéodory theorem

— given measure, μ, on an algebra, $\mathscr{A}$ and outer measure, μ^* induced by μ

- $E \subset X$ is μ^*-measurable *if and only if* exist $A \in \mathscr{A}_{\sigma\delta}$ and $B \subset X$ with $\mu^* B = 0$ such that
$$E = A \sim B$$
  - for $B \subset X$ with $\mu^* B = 0$, exists $C \in \mathscr{A}_{\sigma\delta}$ with $\mu^* C = 0$ such that $B \subset C$

- *Carathéodory theorem -* restriction, $\bar{\mu}$, of μ^* to μ^*-measurable sets if extension of μ to σ-algebra containing $\mathscr{A}$
  - if μ is finite or σ-finite, so is $\bar{\mu}$ respectively
  - if μ is σ-finite, $\bar{\mu}$ is only measure on smallest σ-algebra containing $\mathscr{A}$ which is extension of μ

Product measures

- for countable disjoint collection of measurable rectangles, $\langle(A_n \times B_n)\rangle$, whose union is measurable rectangle, $A \times B$

$$\lambda(A \times B) = \sum \lambda(A_n \times B_n)$$

- for $x \in X$ and $E \in \mathscr{R}_{\sigma\delta}$

$$E_x = \{y | \langle x, y\rangle \in E\}$$

  is measurable subset of Y
- for $E \subset \mathscr{R}_{\sigma\delta}$ with $\mu \times \nu(E) < \infty$, function, g, defined by

$$g(x) = \nu E_x$$

  is measurable function of x and

$$\int g d\mu = \mu \times \nu(E)$$

- XXX

Carathéodory outer measures

- set, X, of points and set, Γ, of real-valued functions on X
- two sets for which exist $a > b$ such that function, φ, greater than a on one set and less than b on the other set, said to be *separated by function, φ*
- outer measure, μ^*, with $(\forall A, B \subset X$ separated by $f \in \Gamma)(\mu^*(A \cup B) = \mu^* A + \mu^* B)$, called *Carathéodory outer measure with respect to Γ*
- outer measure, μ^*, on metric space, $\langle X, \rho, , \rangle$ for which $\mu^*(A \cup B) = \mu^* A + \mu^* B$ for $A, B \subset X$ with $\rho(A, B) > 0$, called *Carathéodory outer measure for X* or *metric outer measure*
- for *Carathéodory outer measure, μ^*, with respect to Γ*, every function in Γ is μ^*-measurable
- for *Carathéodory outer measure, μ^*, for metric space, $\langle X, \rho, \rangle$*, every closed set (hence every Borel set) is measurable with respect to μ^*

References

References

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